SKEW PRODUCTS AND CROSSED PRODUCTS BY COACTIONS

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Abstract. Given a labeling c of the edges of a directed graph E by elements of a discrete group G, one can form a skew-product graph $E \times_c G$. We show, using the universal properties of the various constructions involved, that there is a coaction δ of G on $C^*(E)$ such that $C^*(E \times_c G)$ is isomorphic to the crossed product $C^*(E) \times_\delta G$. This isomorphism is equivariant for the dual action $\widehat{\delta}$ and a natural action γ of G on $C^*(E \times_c G)$; following results of Kumjian and Pask, we show that

$$C^*(E \times_c G) \times_\gamma G \cong C^*(E \times_c G) \times_{\gamma,r} G \cong C^*(E) \otimes \mathcal{K}(\ell^2(G)),$$

and it turns out that the action γ is always amenable. We also obtain corresponding results for r-discrete groupoids Q and continuous homomorphisms $c\colon Q \to G$, provided Q is amenable. Some of these hold under a more general technical condition which obtains whenever Q is amenable or second-countable.

1. Introduction

The C^*-algebra of a directed graph E is the universal C^*-algebra $C^*(E)$ generated by a family of partial isometries which are parameterized by the edges of the graph and satisfy relations of Cuntz-Krieger type reflecting the structure of the graph. A labeling c of the edges by elements of a discrete group G gives rise to a skew-product graph $E \times_c G$, and the natural action of G by translation on $E \times_c G$ lifts to an action γ of G by automorphisms of $C^*(E \times_c G)$. Kumjian and Pask have recently proved ([8, Corollary 3.9]) that

$$C^*(E \times_c G) \times_\gamma G \cong C^*(E) \otimes \mathcal{K}(\ell^2(G)). \tag{1.1}$$

From this they obtained an elegant description of the crossed product $C^*(F) \times_\beta G$ arising from a free action of G on a graph F ([8, Corollary 3.10]).

Kumjian and Pask studied $C^*(E \times_c G)$ by observing that the groupoid model for $E \times_c G$ is a skew product of the groupoid model for E, and establishing an analogous stable isomorphism for the C^*-algebras of skew-product groupoids. They also mentioned that one could obtain these stable isomorphisms from duality theory and a result of Masuda (see [8, Note 3.7]). This second argument raises some interesting issues, which are settled in this paper.

We begin in Section 2 by tackling graph C^*-algebras directly. We show that $C^*(E \times_c G)$ can be realized as the crossed product $C^*(E) \times_\delta G$ by a coaction δ of G (see Theorem 2.4),

1991 *Mathematics Subject Classification.* Primary 46L55.

Key words and phrases. C^*-algebra, coaction, skew product, directed graph, groupoid, duality.

Research partially supported by National Science Foundation Grant DMS9401253 and the Australian Research Council.

and apply the duality theorem of Katayama [7] to deduce that

$$C^*(E \times_c G) \times_{\gamma,r} G \cong (C^*(E) \times_\delta G) \times_{\hat{\delta},r} G \cong C^*(E) \otimes \mathcal{K}(\ell^2(G)) \tag{1.2}$$

(see Corollary 2.5). Since Katayama's theorem involves the reduced crossed product, the result in (1.2) is slightly different from Kumjian and Pask's (1.1) concerning full crossed products. Together, these two results suggest that the action of G on $C^*(E \times_c G)$ should be amenable; we prove this in Section 3 by giving a new proof of the Kumjian-Pask theorem which allows us to see directly that the regular representation of $C^*(E \times_c G) \times_\gamma G$ is faithful.

Our proof of the Kumjian-Pask theorem is elementary in the sense that it uses only the universal properties of graph C^*-algebras, and avoids groupoid and other models. It is therefore slightly more general, and will appeal to those who are primarily interested in graph C^*-algebras. Aficionados of groupoids, however, will naturally ask if we can produce similar results for the C^*-algebra $C^*(Q \times_c G)$ of a skew-product groupoid $Q \times_c G$. We do this (at least for r-discrete groupoids Q) in the second half of the paper.

Masuda has already identified the groupoid algebra $C^*(Q \times_c G)$ as a crossed product by a coaction, in the context of spatially-defined groupoid C^*-algebras, coactions and crossed products ([11, Theorem 3.2]). Nowadays, one would prefer to use full coactions and crossed products, and to give arguments which exploit their universal properties. The result we obtain this way, Theorem 4.3, is more general than could be deduced from [11], and highlights an intriguing technical question: does the C^*-algebra of the subgroupoid $c^{-1}(e) = \{q \in Q \mid c(q) = e\}$ embed faithfully in $C^*(Q)$? We answer this in the affirmative for Q amenable (Lemma 5.8) or second countable (Theorem 6.2).

In Section 5, we establish the amenability of the canonical action of G on $C^*(Q \times_c G)$ when Q is amenable. The results of Section 5 are analogous to those of Section 3, but here we show directly that the action is amenable (Proposition 5.6) using the theory of [2] and [12], and deduce the original version of [8, Theorem 3.7] for full crossed products.

Conventions. A *directed graph* is a quadruple $E = (E^0, E^1, r, s)$ consisting of a set E^0 of vertices, a set E^1 of edges and maps $r, s\colon E^1 \to E^0$ describing the range and source of edges. (This notation is becoming standard because one can then write E^n for the set of paths of length n, and think of vertices as paths of length 0.) The graph E is *row-finite* if each vertex emits at most finitely many edges. Our graphs may have sources and sinks.

All groups in this paper are discrete. A *coaction* of a group G on a C^*-algebra A is an injective nondegenerate homomorphism δ of A into the spatial tensor product $A \otimes C^*(G)$ such that $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta_G) \circ \delta$. The *crossed product* $A \times_\delta G$ is the universal C^*-algebra generated by a covariant representation (j_A, j_G) of (A, G, δ). In general, we use the conventions of [15]. We shall write λ for the left regular representation of a group G on $\ell^2(G)$, ρ for the right regular representation, and M for the representation of $C_0(G)$ by multiplication operators on $\ell^2(G)$. The characteristic function of a set K will be denoted by χ_K.

2. Skew-product graphs and duality

Let $E = (E^0, E^1, r, s)$ be a row-finite directed graph. Following [9], a *Cuntz-Krieger E-family* is a collection $\{t_f, q_v \mid f \in E^1, v \in E^0\}$ of partial isometries t_f and mutually

orthogonal projections q_v in a C^*-algebra B such that

$$t_f^* t_f = q_{r(f)} \quad \text{and} \quad q_v = \sum_{s(f)=v} t_f t_f^*$$

for each $f \in E^1$ and every $v \in E^0$ which is not a sink. By [9, Theorem 1.2], there is an essentially unique C^*-algebra $C^*(E)$, generated by a Cuntz-Krieger E-family $\{ s_f, p_v \}$, which is universal in the sense that for any Cuntz-Krieger E-family $\{ t_f, q_v \}$ in a C^*-algebra B, there is a homomorphism $\Phi = \Phi_{t,q} \colon C^*(E) \to B$ such that $\Phi(s_f) = t_f$ and $\Phi(p_v) = q_v$ for $f \in E^1, v \in E^0$. If $\sum_v q_v \to 1$ strictly in $M(B)$, we say that $\{ t_f, q_v \}$ is a *nondegenerate* Cuntz-Krieger E-family, and the homomorphism $\Phi_{t,q}$ is then nondegenerate. Products $s_e^* s_f$ cancel (see [9, Lemma 1.1]), so $C^*(E)$ is densely spanned by the projections p_v and products of the form $s_\mu s_\nu^* = s_{e_1} s_{e_2} \dots s_{e_n} s_{f_m}^* \dots s_{f_1}^*$, where μ and ν are finite paths in the graph E.

For each $z \in \mathbb{T}$, $\{ z s_f, p_v \}$ is a Cuntz-Krieger E-family, so there is an automorphism α_z of $C^*(E)$ such that $\alpha_z(s_f) = z s_f$ and $\alpha_z(p_v) = p_v$. For each pair of paths μ, ν the map $z \mapsto \alpha_z(s_\mu s_\nu^*)$ is continuous, and it follows from a routine $\epsilon/3$-argument that α is a strongly continuous action of $\mathbb{T}$ on $C^*(E)$. It was observed in [6] that the existence of this *gauge action* α characterizes the universal C^*-algebra $C^*(E)$. The following extension of [6, Theorem 2.3] will appear in [3]; it is proved by modifying the proof of [6, Theorem 2.3] to allow for infinite graphs and the possibility of sinks.

Lemma 2.1. *Let E be a row-finite directed graph, and suppose B is a C^*-algebra generated by a Cuntz-Krieger E-family $\{ t_f, q_v \}$. If all the t_f and q_v are non-zero and there is a strongly continuous action β of $\mathbb{T}$ on B such that $\beta_z(t_f) = z t_f$ and $\beta_z(q_v) = q_v$, then the canonical homomorphism $\Phi_{t,q} \colon C^*(E) \to B$ is an isomorphism.*

A *labeling* of E by a group G is just a function $c \colon E^1 \to G$. The *skew-product* graph $E \times_c G$ is the directed graph with vertex set $E^0 \times G$, edge set $E^1 \times G$, and range and source maps defined by

$$r(f,t) = (r(f), t) \quad \text{and} \quad s(f,t) = (s(f), c(f)t) \qquad \text{for } (f,t) \in E^1 \times G.$$

Since $s^{-1}(v,t) = \{ (f, c(f)t) \mid f \in s^{-1}(v) \}$, the vertex $(v,t) \in (E \times_c G)^0$ emits the same number of edges as $v \in E^0$; thus $E \times_c G$ is row-finite if and only if E is, and (v,t) is a sink if and only if v is.

Remark 2.2. Our skew product $E \times_c G$ is not quite the same as the versions $E(c)$ in [8] and E^c in [5]; however, there are isomorphisms $\phi \colon E(c) \to E \times_c G$ and $\psi \colon E^c \to E \times_c G$ given by

$$\phi(t,v) = \psi(v,t) = (v, t^{-1}) \quad \text{and} \quad \phi(t,f) = \psi(f,t) = (f, c(f)^{-1} t^{-1}).$$

Our conventions were chosen to make the isomorphism of Theorem 2.4 more natural.

Lemma 2.3. *Let c be a labeling of a row-finite directed graph E by a discrete group G. Then there is a coaction δ of G on $C^*(E)$ such that*

$$\delta(s_f) = s_f \otimes c(f) \quad \textit{and} \quad \delta(p_v) = p_v \otimes 1 \qquad \textit{for } f \in E^1,\ v \in E^0. \tag{2.1}$$

Proof. Straightforward calculations show that $\{ s_f \otimes c(f), p_v \otimes 1 \}$ is a nondegenerate Cuntz-Krieger E-family, so the universal property gives a nondegenerate homomorphism $\delta\colon C^*(E) \to C^*(E) \otimes C^*(G)$ which satisfies (2.1). Lemma 2.1 implies that δ is injective: take $\beta = \alpha \otimes \mathrm{id}$, where α is the gauge action of $\mathbb{T}$ on $C^*(E)$. It follows from (2.1) that the coaction identity $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta_G) \circ \delta$ holds on generators s_f and p_v, and it extends by algebra and continuity to all of $C^*(E)$. □

The group G acts on the graph $E \times_c G$ by right translation, so that $t \cdot (v, s) = (v, st^{-1})$ and $t \cdot (f, s) = (f, st^{-1})$; this induces an action $\gamma : G \to \operatorname{Aut} C^*(E \times_c G)$ such that

$$\gamma_t(s_{(f,s)}) = s_{(f,st^{-1})} \quad \text{and} \quad \gamma_t(p_{(v,s)}) = p_{(v,st^{-1})}. \tag{2.2}$$

Theorem 2.4. *Let c be a labeling of a row-finite directed graph E by a discrete group G, and let δ be the coaction from Lemma 2.3. Then*

$$C^*(E) \times_\delta G \cong C^*(E \times_c G),$$

equivariantly for the dual action $\widehat{\delta}$ and the action γ of Equation (2.2).

Proof. We use the calculus of [4] to handle elements of the crossed product $C^*(E) \times_\delta G$. For each $t \in G$, let $C^*(E)_t = \{ a \in C^*(E) \mid \delta(a) = a \otimes t \}$ denote the corresponding spectral subspace; we write a_t to denote a generic element of $C^*(E)_t$. (This subscript convention conflicts with the standard notation for Cuntz-Krieger families: each partial isometry s_f is in $C^*(E)_{c(f)}$, and each projection p_v is in $C^*(E)_e$, where e is the identity element of G. We hope this does not cause confusion.) Then $C^*(E) \times_\delta G$ is densely spanned by the set $\{ (a_t, u) \mid a_t \in C^*(E)_t;\ t, u \in G \}$, and the algebraic operations are given on this set by

$$(a_r, s)(a_t, u) = (a_r a_t, u) \quad \text{if } s = tu \text{ (and 0 if not), and} \quad (a_t, u)^* = (a_t^*, tu).$$

(If $(j_{C^*(E)}, j_G)$ is the canonical covariant homomorphism of $(C^*(E), C_0(G))$ into $M(C^*(E) \times_\delta G)$, then (a_t, u) is by definition $j_{C^*(E)}(a_t) j_G(\chi_{\{u\}})$.) The dual action $\widehat{\delta}$ of G on $C^*(E) \times_\delta G$ is characterized by $\widehat{\delta}_s(a_t, u) = (a_t, us^{-1})$.

We aim to define a Cuntz-Krieger $E \times_c G$-family $\{t_{(f,t)}, q_{(v,t)}\}$ in $C^*(E) \times_\delta G$ by putting

$$t_{(f,t)} = (s_f, t) \quad \text{and} \quad q_{(v,t)} = (p_v, t)$$

for $(f, t) \in (E \times_c G)^0$ and $(v, t) \in (E \times_c G)^1$. To see that this is indeed a Cuntz-Krieger family, note first that $p_v \in C^*(E)_e$ for all vertices v, so the $q_{(v,t)}$ are mutually orthogonal projections. Next note that $s_f \in C^*(E)_{c(f)}$, so

$$t^*_{(f,t)} t_{(f,t)} = (s_f^*, c(f)t)(s_f, t) = (s_f^* s_f, t) = (p_{r(f)}, t) = q_{r(f,t)};$$

if (v, t) is a not sink, then v is not a sink in E, so

$$\begin{aligned} q_{(v,t)} &= (p_v, t) = \sum_{s(f)=v} (s_f s_f^*, t) = \sum_{s(f)=v} (s_f, c(f)^{-1}t)(s_f^*, t) \\ &= \sum_{s(f)=v} (s_f, c(f)^{-1}t)(s_f, c(f)^{-1}t)^* = \sum_{s(f,r)=(v,t)} t_{(f,r)} t^*_{(f,r)}. \end{aligned}$$

This shows that $\{t_{(f,t)}, q_{(v,t)}\}$ is a Cuntz-Krieger $E \times_c G$-family.

The universal property of the graph algebra now gives a homomorphism $\Phi = \Phi_{t,q}\colon C^*(E \times_c G) \to C^*(E) \times_\delta G$ such that $\Phi(s_{(f,t)}) = t_{(f,t)}$ and $\Phi(p_{(v,t)}) = q_{(v,t)}$;

we shall prove that it is an isomorphism using Lemma 2.1. The gauge action α of $\mathbb{T}$ on $C^*(E)$ commutes with the coaction δ, in the sense that $\delta(\alpha_z(a)) = \alpha_z \otimes \mathrm{id}(\delta(a))$ for each $z \in \mathbb{T}$ and $a \in C^*(E)$; it therefore induces an action $\alpha \times \mathrm{id}$ of $\mathbb{T}$ on $C^*(E) \times_\delta G$ such that

$$(\alpha \times \mathrm{id})_z(t_{(f,t)}) = (\alpha \times \mathrm{id})_z(s_f, t) = (zs_f, t) = zt_{(f,t)} \quad \text{and} \quad (\alpha \times \mathrm{id})_z(q_{(v,t)}) = q_{(v,t)}.$$

One can see that the elements of $\{t_{(f,t)}, q_{(v,t)}\}$ are all non-zero by fixing a faithful representation π of $C^*(E)$ and considering the regular representation $\operatorname{Ind}\pi = ((\pi \otimes \lambda) \circ \delta) \times (1 \otimes M)$ of $C^*(E) \times_\delta G$ induced by π: the operator $\operatorname{Ind}\pi(t_{(f,t)})$, for example, is just $(\pi(s_f) \otimes \lambda_{c(f)})(1 \otimes M(\chi_{\{t\}}))$, which has non-zero initial projection $\pi(p_{r(f)}) \otimes M(\chi_{\{t\}})$. Since $(s_e, c(f)t)(s_f, t) = (s_e s_f, t)$ and $(s_e, c(f)^{-1}t)(s_f^*, t) = (s_e s_f^*, t)$, the range of Φ contains the generating family $\{j_{C^*(E)}(s_\mu s_\nu^*) j_G(\chi_{\{t\}}), j_{C^*(E)}(p_v) j_G(\chi_{\{t\}})\}$, and hence is all of $C^*(E) \times_\delta G$. Thus Lemma 2.1 applies, and Φ is an isomorphism of $C^*(E \times_c G)$ onto $C^*(E) \times_\delta G$.

Finally, we check that Φ intertwines γ and $\widehat{\delta}$:

$$\Phi(\gamma_r(s_{(f,t)})) = \Phi(s_{(f,tr^{-1})}) = (s_f, tr^{-1}) = \widehat{\delta}_r(s_f, t) = \widehat{\delta}_r(\Phi(s_{(f,t)})),$$

and this completes the proof. □

Corollary 2.5. *Let c be a labeling of a row-finite directed graph E by a discrete group G, and let γ be the action of Equation* (2.2)*. Then*

$$C^*(E \times_c G) \times_{\gamma,r} G \cong C^*(E) \otimes \mathcal{K}(\ell^2(G)).$$

Proof. The corollary follows from Theorem 2.4 and Katayama's duality theorem [7, Theorem 8]. (Even though we are using full coactions, Katayama's theorem still applies: the regular representation is an isomorphism of $C^*(E) \times_\delta G$ onto the (reduced) crossed product by the reduction of δ; see [13, Corollary 2.6]). □

3. Skew-product graphs: the full crossed product

Theorem 3.1. *Let c be a labeling of a row-finite directed graph E by a discrete group G, and let γ be the action of G defined by Equation* (2.2)*. Then*

$$C^*(E \times_c G) \times_\gamma G \cong C^*(E) \otimes \mathcal{K}(\ell^2(G)).$$

Proof. Since G is discrete, $C^*(E \times_c G) \times_\gamma G$ is generated by the set of products $\{ s_{(f,r)} u_t,\, p_{(v,r)} u_t \}$, where $\{ s_{(f,r)},\, p_{(v,r)} \}$ is a nondegenerate Cuntz-Krieger $E \times_c G$-family and u is the canonical homomorphism of G into $UM(C^*(E \times_c G) \times_\gamma G)$ satisfying

$$u_t s_{(f,r)} = s_{(f,rt^{-1})} u_t \quad \text{and} \quad u_t p_{(v,r)} = p_{(v,rt^{-1})} u_t \qquad \text{for } t \in G. \tag{3.1}$$

Moreover, the crossed product is universal in the sense that for any nondegenerate Cuntz-Krieger $E \times_c G$-family $\{ t_{(f,r)}, q_{(v,r)} \}$ in a C^*-algebra B and any homomorphism v of G into $UM(B)$ satisfying the analogue of (3.1), there is a unique nondegenerate homomorphism $\Theta = \Theta_{t,q,v}$ of $C^*(E \times_c G) \times_\gamma G$ into B which takes each generator to its counterpart in B.

We now construct such a family $\{ t_{(f,r)}, q_{(v,r)}, v_t \}$ in $C^*(E) \otimes \mathcal{K}(\ell^2(G))$. With $\{s_f, p_v\}$ denoting the canonical generators of $C^*(E)$ and writing χ_r for $M(\chi_{\{r\}})$, we set

$$t_{(f,r)} = s_f \otimes \lambda_{c(f)} \chi_r \quad \text{and} \quad q_{(v,r)} = p_v \otimes \chi_r.$$

Then the $q_{(v,r)}$ are clearly mutually orthogonal projections, and $\sum_{v,r} q_{(v,r)} \to 1$ strictly in $M(C^*(E) \otimes \mathcal{K}(\ell^2(G)))$. Further, we have

$$t^*_{(f,r)} t_{(f,r)} = s_f^* s_f \otimes \chi_r^* \lambda_{c(f)}^* \lambda_{c(f)} \chi_r = s_f^* s_f \otimes \chi_r = p_{r(f)} \otimes \chi_r = q_{r(f,r)},$$

and

$$\begin{aligned} q_{(v,r)} &= \sum_{s(f)=v} s_f s_f^* \otimes \chi_r \\ &= \sum_{s(f)=v} (s_f \otimes \chi_r \lambda_{c(f)})(s_f \otimes \chi_r \lambda_{c(f)})^* \\ &= \sum_{s(f)=v} (s_f \otimes \lambda_{c(f)} \chi_{c(f)^{-1}r})(s_f \otimes \lambda_{c(f)} \chi_{c(f)^{-1}r})^* \\ &= \sum_{s(f)=v} t_{(f,c(f)^{-1}r)} t^*_{(f,c(f)^{-1}r)} \\ &= \sum_{s(f,t)=(v,r)} t_{(f,t)} t^*_{(f,t)}, \end{aligned}$$

so $\{t_{(f,r)}, q_{(v,r)}\}$ is a Cuntz-Krieger $E \times_c G$-family. The unitary elements $1 \otimes \rho_t$ of $M(C^*(E) \otimes \mathcal{K}(\ell^2(G)))$ satisfy

$$\begin{aligned} (1 \otimes \rho_t) t_{(f,r)} = s_f \otimes \rho_t \lambda_{c(f)} \chi_r &= s_f \otimes \lambda_{c(f)} \chi_{rt^{-1}} \rho_t = t_{(f,rt^{-1})}(1 \otimes \rho_t), \quad \text{and} \\ (1 \otimes \rho_t) q_{(v,r)} = p_v \otimes \rho_t \chi_r &= p_v \otimes \chi_{rt^{-1}} \rho_t = q_{(v,rt^{-1})}(1 \otimes \rho_t); \end{aligned}$$

thus we get a nondegenerate homomorphism $\Theta = \Theta_{t,q,1\otimes\rho} \colon C^*(E \times_c G) \times_\gamma G \to C^*(E) \otimes \mathcal{K}(\ell^2(G))$ such that

$$\Theta(s_{(f,r)}) = t_{(f,r)}, \quad \Theta(p_{(v,r)}) = q_{(v,r)}, \quad \text{and} \quad \Theta(u_t) = 1 \otimes \rho_t. \tag{3.2}$$

To construct the inverse for Θ, we use a universal property of $C^*(E) \otimes \mathcal{K}(\ell^2(G))$. Let σ denote the action of G on $C_0(C)$ by right translation: $\sigma_s(f)(t) = f(ts)$. The regular representation $M \times \rho$ is an isomorphism of the crossed product $C_0(G) \times_\sigma G$ onto $\mathcal{K}(\ell^2(G))$, so we can view $\mathcal{K}(\ell^2(G))$ as the universal C^*-algebra generated by the set of products $\{\chi_r \rho_t \mid r, t \in G\}$, where ρ is a unitary homomorphism ρ of G and $\{\chi_r\}$ is a set of mutually orthogonal projections satisfying

$$\rho_t \chi_r = \chi_{rt^{-1}} \rho_t. \tag{3.3}$$

Thus to get a homomorphism defined on $C^*(E) \otimes \mathcal{K}(\ell^2(G))$ we need a Cuntz-Krieger E-family $\{t_f, q_v\}$ and a family $\{y_r, u_t\}$ analogous to $\{\chi_r, \rho_t\}$ which commutes with the Cuntz-Krieger family.

We begin by constructing a family $\{y_r, u_t\}$ in $M(C^*(E \times_c G) \times_\gamma G)$. We claim that, for fixed $r \in G$, the sum $\sum_v p_{(v,r)}$ converges strictly in $M(C^*(E \times_c G) \times_\gamma G)$. Because the canonical embedding $j_{C^*(E \times_c G)}$ has a strictly continuous extension, it is enough to check that the sum converges strictly in $M(C^*(E \times_c G))$. Because all the finite sums are projections, they have norm uniformly bounded by 1, and it is enough to check that $\left(\sum_v p_{(v,r)}\right) s_\mu s_\nu^*$ and $s_\mu s_\nu^* \left(\sum_v p_{(v,r)}\right)$ converge for each pair of paths μ, ν in $E \times_c G$; and that $\left(\sum_v p_{(v,r)}\right) p_{(u,t)}$ and $p_{(u,t)} \left(\sum_v p_{(v,r)}\right)$ converge for each vertex (u,t) in $E \times_c G$. But

in each case these sums reduce to a single term, so this is trivially true. Thus we may put $y_r = \sum_v p_{(v,r)} \in M(C^*(E \times_c G) \times_\gamma G)$.

Now $\{ y_r \mid r \in G \}$ is a mutually orthogonal family of projections, and $\sum_{v,r} p_{(v,r)} \to 1$ strictly in $M(C^*(E \times_c G))$, so $\sum_s y_s \to 1$ strictly in $M(C^*(E \times_c G) \times_\gamma G)$. Moreover, if u is the canonical homomorphism of G into $M(C^*(E \times_c G) \times_\gamma G)$, then

$$u_t y_r = u_t \sum_v p_{(v,r)} = \sum_v p_{(v,rt^{-1})} u_t = y_{rt^{-1}} u_t;$$

thus the family $\{ y_r, u_t \}$ satisfies the analogue of (3.3), and therefore gives a nondegenerate homomorphism $y \times u \colon \mathcal{K}(\ell^2(G)) \to M(C^*(E \times_c G) \times_\gamma G)$. This homomorphism extends to $\mathcal{B}(\ell^2(G)) = M(\mathcal{K}(\ell^2(G)))$, and we can define unitaries $w_t = y \times u(\lambda_t)$ which satisfy $w_t y_r = y_{tr} w_t$ and $w_t u_r = u_r w_t$ for each $r, t \in G$.

Arguing as for the y_r shows that, for each fixed v and f, the sums $\sum_r p_{(v,r)}$ and $\sum_r s_{(f,r)}$ converge strictly in $M(C^*(E \times_c G))$. Thus we may define t_f and q_v in $M(C^*(E \times_c G) \times_\gamma G)$ by

$$t_f = \Big(\sum_r s_{(f,r)} \Big) w_{c(f)^{-1}} \quad \text{and} \quad q_v = \sum_r p_{(v,r)}.$$

Now $\{q_v\}$ is a family of mutually orthogonal projections; to check the Cuntz-Krieger relations for $\{ t_f, q_v \}$, first note that

$$\Big(\sum_r s_{(f,r)} \Big)^* \Big(\sum_t s_{(f,t)} \Big) = \sum_{r,t} s^*_{(f,r)} s_{(f,t)} = \sum_r s^*_{(f,r)} s_{(f,r)} = \sum_r p_{r(f,r)} = q_{r(f)},$$

so that

$$t^*_f t_f = w_{c(f)} \Big(\sum_r s_{(f,r)} \Big)^* \Big(\sum_t s_{(f,t)} \Big) w^*_{c(f)} = w_{c(f)} q_{r(f)} w^*_{c(f)}. \tag{3.4}$$

Easy calculations show that $y_t q_v = q_v y_t$ and $u_t q_v = q_v u_t$, so each q_v commutes with everything in the range of $y \times u$ in $M(C^*(E \times_c G) \times_\gamma G)$, and in particular with each w_t; thus Equation (3.4) implies that $t^*_f t_f = q_{r(f)}$. We also have

$$\begin{aligned} q_v &= \sum_r \sum_{s(f,t)=(v,r)} s_{(f,t)} s^*_{(f,t)} \\ &= \sum_r \sum_{s(f)=v} s_{(f,c(f)^{-1}r)} s^*_{(f,c(f)^{-1}r)} \\ &= \sum_{s(f)=v} \sum_r s_{(f,r)} s^*_{(f,r)} \\ &= \sum_{s(f)=v} \Big(\sum_r s_{(f,r)} w_{c(f)^{-1}} \Big) \Big(\sum_t s_{(f,t)} w_{c(f)^{-1}} \Big)^* \\ &= \sum_{s(f)=t} t_f t^*_f, \end{aligned}$$

and $\sum_v q_v = \sum_{v,r} p_{(v,r)} \to 1$ strictly in $M(C^*(E \times_c G) \times_\gamma G)$, so the set $\{ t_f, q_v \}$ is a nondegenerate Cuntz-Krieger E-family.

We have already observed that each q_v commutes with the range of $y \times u$. Further calculations show that

$$\begin{aligned} y_s t_f &= \Big(\sum_v p_{(v,s)}\Big)\Big(\sum_r s_{(f,r)}\Big) w_{c(f)^{-1}} \\ &= s_{(f,c(f)^{-1}s)} w_{c(f)^{-1}} \\ &= \Big(\sum_r s_{(f,r)}\Big)\Big(\sum_v p_{(v,c(f)^{-1}s)}\Big) w_{c(f)^{-1}} \\ &= \Big(\sum_r s_{(f,r)}\Big) y_{c(f)^{-1}s} w_{c(f)^{-1}} \\ &= \Big(\sum_r s_{(f,r)}\Big) w_{c(f)^{-1}} y_s \\ &= t_f y_s \end{aligned} \tag{3.5}$$

and

$$u_t t_f = u_t \Big(\sum_r s_{(f,r)}\Big) w_{c(f)^{-1}} = \Big(\sum_r s_{(f,rs^{-1})}\Big) u_t w_{c(f)^{-1}} = \Big(\sum_r s_{(f,r)}\Big) w_{c(f)^{-1}} u_t = t_f u_t.$$

Thus the homomorphisms $\Phi_{t,q}$ of $C^*(E)$ and $y \times u$ of $\mathcal{K}(\ell^2(G))$ into $M(C^*(E \times_c G) \times_\gamma G)$ have commuting ranges, and combine to give a homomorphism Υ of $C^*(E) \otimes \mathcal{K}(\ell^2(G))$ into $M(C^*(E \times_c G) \times_\gamma G)$ such that $\Upsilon(s_f \otimes 1) = t_f$, $\Upsilon(p_v \otimes 1) = q_v$, $\Upsilon(1 \otimes \chi_r) = y_r$, and $\Upsilon(1 \otimes \rho_t) = u_t$. From (3.5) we deduce that

$$\Upsilon(s_f \otimes \chi_r \rho_t) = t_f y_r u_t = s_{(f,c(f)^{-1}r)} w_{c(f)^{-1}} u_t = s_{(f,c(f)^{-1}r)} u_t w_{c(f)^{-1}}; \tag{3.6}$$

since this and $\Upsilon(p_v \otimes \chi_r \rho_t) = q_v y_r u_t = p_{(v,r)} u_t$ belong to $C^*(E \times_c G) \times_\gamma G$, it follows that Υ maps $C^*(E) \otimes \mathcal{K}(\ell^2(G))$ into $C^*(E \times_c G) \times_\gamma G$.

We shall show that Θ and Υ are inverses of one another by checking that $\Upsilon \circ \Theta$ is the identity on the generating set $\{s_{(f,r)} u_t,\ p_{(v,r)} u_t\}$ for $C^*(E \times_c G) \times_\gamma G$, and that $\Theta \circ \Upsilon$ is the identity on a generating set for $C^*(E) \otimes \mathcal{K}(\ell^2(G))$. First we note that $T \mapsto \Upsilon(1 \otimes T)$ is just $y \times u$ on products $\chi_r \rho_t \in \mathcal{K}(\ell^2(G))$, so $\Upsilon(1 \otimes \lambda_t) = w_t$ by definition of w_t. And since Θ extends to a strictly continuous map on $M(C^*(E \times_c G) \times_\gamma G)$, we have

$$\Theta(y_r u_t) = \Theta\Big(\sum_v p_{(v,r)} u_t\Big) = \sum_v p_v \otimes \chi_r \rho_t = 1 \otimes \chi_r \rho_t,$$

which implies that $\Theta(w_t) = 1 \otimes \lambda_t$ for $t \in G$.

We can now compute:

$$\begin{aligned} \Upsilon \circ \Theta(s_{(f,s)} u_t) &= \Upsilon(s_f \otimes \lambda_{c(f)} \chi_s \rho_t) \\ &= \Big(\sum_r s_{(f,r)}\Big) w_{c(f)^{-1}} w_{c(f)} \Big(\sum_v p_{(v,s)}\Big) u_t \end{aligned}$$

$$\begin{aligned} &= \Big(\sum_r s_{(f,r)}\Big)\Big(\sum_v p_{(v,s)}\Big)u_t \\ &= s_{(f,s)}u_t \end{aligned}$$

and

$$\Upsilon\circ\Theta(p_{(v,s)}u_t) = \Upsilon(p_v\otimes\chi_s\rho_t) = \Big(\sum_r p_{(v,r)}\Big)\Big(\sum_w p_{(w,s)}\Big)u_t = p_{(v,s)}u_t,$$

which shows that $\Upsilon\circ\Theta$ is the identity. Using (3.6) gives

$$\begin{aligned} \Theta\circ\Upsilon(s_f\otimes\chi_r\rho_t) &= \Theta(s_{(f,c(f)^{-1}r)}u_t w_{c(f)^{-1}}) \\ &= s_f\otimes\lambda_{c(f)}\chi_{c(f)^{-1}r}\lambda_{c(f)^{-1}}\rho_t \\ &= s_f\otimes\chi_r\rho_t \end{aligned}$$

and

$$\Theta\circ\Upsilon(p_v\otimes\chi_r\rho_t) = \Theta(p_{(v,r)}u_t) = p_v\otimes\chi_r\rho_t,$$

which shows that $\Theta\circ\Upsilon$ is the identity. □

Theorem 3.1 and Corollary 2.5 imply that $C^*(E\times_c G)\times_\gamma G$ and $C^*(E\times_c G)\times_{\gamma,r} G$ are isomorphic C^*-algebras, so it is natural to ask if the action γ is amenable in the sense that the regular representation of the crossed product is faithful. To see that it is, consider the following diagram:

$$\tag{3.7}\begin{tikzcd} C^*(E\times_c G)\times_\gamma G \arrow[rr, "\text{Theorem 2.4}"] \arrow[dr, "\text{Theorem 3.1}"] \arrow[dd, "\text{regular representation}"'] & & C^*(E)\times_\delta G\times_{\widehat{\delta}} G \arrow[dd, "\text{regular representation}"] \\ & C^*(E)\otimes\mathcal{K} & \\ C^*(E\times_c G)\times_{\gamma,r} G \arrow[rr, "\text{Theorem 2.4}"'] & & C^*(E)\times_\delta G\times_{\widehat{\delta},r} G. \arrow[ul, "\text{Katayama}"'] \end{tikzcd}$$

Let $(j_{C^*(E)}, j_G)\colon (C^*(E), C_0(G)) \to M(C^*(E)\times_\delta G)$ and $(i_{C^*(E)\times G}, i_G)\colon (C^*(E)\times_\delta G, G) \to M(C^*(E)\times_\delta G\times_{\widehat{\delta}} G)$ be the canonical maps. Inspection of the formulas on page 768 of [10] shows that composing the regular representation of $C^*(E)\times_\delta G\times_{\widehat{\delta}} G$ with Katayama's isomorphism ([7, Theorem 8]) gives

$$i_{C^*(E)\times G}(j_{C^*(E)}(a)) \mapsto \operatorname{id}\otimes\lambda(\delta(a)),\ i_{C^*(E)\times G}(j_G(g)) \mapsto 1\otimes M(g),\ i_G(t)\mapsto 1\otimes\rho_t$$

for $a\in C^*(E)$, $g\in C_c(G)$, and $t\in G$. Thus chasing generators in $C^*(E\times_c G)\times_\gamma G$ round the outside of the upper right-hand triangle in Diagram (3.7) yields

$$\begin{aligned} s_{(f,r)} &\mapsto i_{C^*(E)\times G}(j_{C^*(E)}(s_f)j_G(\chi_r)) \mapsto \operatorname{id}\otimes\lambda(\delta(s_f))1\otimes\chi_r = t_{(f,r)}, \\ p_{(v,r)} &\mapsto i_{C^*(E)\times G}(j_{C^*(E)}(p_v)j_G(\chi_r)) \mapsto \operatorname{id}\otimes\lambda(\delta(p_v))1\otimes\chi_r = q_{(v,r)}, \\ u_t &\mapsto i_G(t)\mapsto 1\otimes\rho_t. \end{aligned}$$

Since this is exactly what the isomorphism Θ from Theorem 3.1 does (see Equation (3.2)), the upper right-hand corner of Diagram (3.7) commutes. But the outside rectangle commutes by general nonsense, so the lower left-hand corner commutes too. This proves:

Corollary 3.2. *Let c be a labeling of a row-finite directed graph E by a discrete group G. Then the action γ of G from Equation* (2.2) *is amenable in the sense that the regular representation of $C^*(E \times_c G) \times_\gamma G$ is faithful.*

Corollary 3.3. *Let G be a discrete group acting freely on a row-finite directed graph F, and let β be the action of G on $C^*(F)$ determined by $\beta_t(s_f) = s_{t\cdot f}$ and $\beta_t(p_v) = p_{t\cdot v}$. Then the regular representation of $C^*(F) \times_\beta G$ is faithful, and*

$$C^*(F) \times_\beta G \cong C^*(F) \times_{\beta,r} G \cong C^*(F/G) \otimes \mathcal{K}(\ell^2(G)).$$

Proof. Since G acts freely, there is a labeling $c\colon (F/G)^1 \to G$ and an isomorphism of F onto $(F/G) \times_c G$ which carries the given action to the action of G by right translation ([5, Theorem 2.2.2]). Thus this corollary follows by applying Corollaries 2.5 and 3.2 to $E = F/G$. □

4. Skew-product groupoids and duality

We will now give groupoid versions of the results in Section 2. Throughout, we consider a discrete group G, and a groupoid Q which is r-discrete in the modern sense that the range map r is a local homeomorphism (so that counting measures on the sets $Q^u = r^{-1}(u)$ for u in the unit space Q^0 give a Haar system on Q). In several of the following arguments, we use the fact that the C^*-algebra of an r-discrete groupoid Q is the enveloping C^*-algebra of $C_c(Q)$; this follows from [16, Theorems 7.1 and 8.1].

Let $c\colon Q \to G$ be a continuous homomorphism. The *skew-product groupoid* $Q \times_c G$ is the set $Q \times G$ with the product topology and operations given for $(x, y) \in Q^2$ and $s \in G$ by

$$(x, c(y)s)(y, s) = (xy, s) \quad \text{and} \quad (x, s)^{-1} = (x^{-1}, c(x)s).$$

Since the range map on the skew-product groupoid is thus given by $r(x, s) = (r(x), c(x)s)$, $Q \times_c G$ is r-discrete whenever Q is. The formula

$$s \cdot (x, t) = (x, ts^{-1}) \qquad \text{for } s, t \in G,\ x \in Q \tag{4.1}$$

defines an action of G by automorphisms of the topological groupoid $Q \times_c G$. We let β denote the induced action on $C^*(Q \times_c G)$, which satisfies

$$\beta_s(f)(x, t) = f\big(s^{-1} \cdot (x, t)\big) = f(x, ts) \qquad \text{for } s, t \in G,\ f \in C_c(Q \times_c G),\ x \in Q. \tag{4.2}$$

Remark 4.1. It is easily checked that the map $(x, s) \mapsto (x, c(x)^{-1}s^{-1})$ gives a topological isomorphism of Renault's skew product [18, Definition I.1.6] onto ours, which transports Renault's action (also used in [8, Proposition 3.7]) into β. Our conventions were chosen to make the isomorphism of Theorem 4.3 more natural.

For $s \in G$ define

$$C_s = \{f \in C_c(Q) \mid \operatorname{supp} f \subseteq c^{-1}(s)\}, \tag{4.3}$$

and put $\mathcal{C} = \bigcup_{s\in G} C_s$. Then with the operations from $C_c(Q)$, $\mathcal{C}$ becomes a *-algebraic bundle (with incomplete fibers) over G in the sense that $C_sC_t \subseteq C_{st}$ and $C_s^* = C_{s^{-1}}$. Since Q is the disjoint union of the open sets $\{c^{-1}(s)\}_{s\in G}$, we have $\operatorname{span}_{s\in G} C_s = C_c(Q)$, which we identify with the space $\Gamma_c(\mathcal{C})$ of finitely supported sections of $\mathcal{C}$.

Lemma 4.2. *Let c be a continuous homomorphism of an r-discrete Hausdorff groupoid Q into a discrete group G. Then there is a coaction δ of G on $C^*(Q)$ such that*

$$\delta(f_s) = f_s \otimes s \qquad \textit{for } s \in G,\ f_s \in C_s.$$

Proof. The above formula extends uniquely to a *-homomorphism of $C_c(Q)$ into $C^*(Q) \otimes C^*(G)$. Since $C^*(Q)$ is the enveloping C^*-algebra of $C_c(Q)$, δ further extends uniquely to a homomorphism of $C^*(Q)$ into $C^*(Q) \otimes C^*(G)$. The coaction identity obviously holds on the generators (that is, the elements of the bundle $\mathcal{C}$), hence on all of $C^*(Q)$. The homomorphism δ is nondegenerate, that is,

$$\overline{\operatorname{span}}\{\delta(C^*(Q))(C^*(Q) \otimes C^*(G))\} = C^*(Q) \otimes C^*(G),$$

since $\delta(f_s)(1 \otimes s^{-1}t) = f_s \otimes t$. To see that δ is injective, let 1_G denote the trivial one-dimensional representation of G, and check on the generators that $(\mathrm{id} \otimes 1_G) \circ \delta = \mathrm{id}$. □

Let $N = c^{-1}(e)$ be the kernel of the homomorphism c, which is an open subgroupoid of Q. Since the restriction of a Haar system to an open subgroupoid gives a Haar system, counting measures give a Haar system on N, so N is an r-discrete groupoid. The inclusion of $C_c(N)$ in $C_c(Q)$ extends to the enveloping C^*-algebras to give a natural homomorphism i of $C^*(N)$ into $C^*(Q)$. For our next results we will need to require that i be faithful. We have been unable to show that this holds in general, although it does hold when Q is amenable (Lemma 5.8), and when Q is second countable (Theorem 6.2).

Theorem 4.3. *Let c be a continuous homomorphism of an r-discrete Hausdorff groupoid Q into a discrete group G, let $N = c^{-1}(e)$, and let δ be the coaction from Lemma* 4.2. *Assume that the natural map $i\colon C^*(N) \to C^*(Q)$ is faithful. Then*

$$C^*(Q) \times_\delta G \cong C^*(Q \times_c G),$$

equivariantly for the dual action $\widehat{\delta}$ and the action β of Equation (4.2).

Proof. Let $\mathcal{C}$ be the *-algebraic bundle over G defined by Equation (4.3), let $\mathcal{C} \times G$ be the product bundle over $G \times G$ whose fiber over (s, t) is $C_s \times \{t\}$, and give $\mathcal{C} \times G$ the algebraic operations

$$(f_s, tu)(g_t, u) = (f_sg_t, u) \quad \text{and} \quad (f_s, t)^* = (f_s^*, st).$$

Then the space $\Gamma_c(\mathcal{C} \times G)$ of finitely supported sections becomes a *-algebra, which can be identified with a dense *-subalgebra of the crossed product $C^*(Q) \times_\delta G$; the dual action is characterized by $\widehat{\delta}_s(f, t) = (f, ts^{-1})$, for $s, t \in G$ and $f \in \mathcal{C}$.

We claim that $C^*(Q) \times_\delta G$ is the enveloping C^*-algebra of $\Gamma_c(\mathcal{C} \times G)$. Since $C^*(Q)$ is the enveloping C^*-algebra of $\Gamma_c(\mathcal{C}) = C_c(Q)$, by [4, Theorem 3.3] it suffices to show that the unit fiber algebra $C^*(Q)_e = \{\, f \in C^*(Q) \mid \delta(f) = f \otimes e \,\}$ of the Fell bundle associated to δ is the enveloping C^*-algebra of C_e. To see this, first note that $C^*(Q)_e$ is the closure of C_e in $C^*(Q)$, which in turn is just $i(C^*(N))$ because i maps $C_c(N)$ onto C_e. But $C^*(N)$ is the enveloping C^*-algebra of $C_c(N)$. Since i is assumed to be faithful, it follows that

$C^*(Q)_e = i(C^*(N))$ is the enveloping C^*-algebra of $C_e = i(C_c(N))$, and this proves the claim.

Now for each $s,t \in G$ put $D_{s,t} = \{f \in C_c(Q \times_c G) \mid \operatorname{supp} f \subseteq c^{-1}(s) \times \{t\}\}$, so $C_c(Q \times_c G) = \operatorname{span}_{s,t\in G} D_{s,t}$. For $f \in \mathcal{C}$ and $t \in G$ define $\Psi(f,t) \in C_c(Q \times_c G)$ by

$$\Psi(f,t)(x,u) = f(x) \qquad \text{if } t = u \text{ (and 0 if not).} \tag{4.4}$$

Then Ψ extends uniquely to a linear bijection Ψ of $\Gamma_c(\mathcal{C} \times G)$ onto $C_c(Q \times_c G)$, since it gives a linear bijection of each fiber $C_s \times \{t\}$ onto the corresponding fiber $D_{s,t}$. In fact, Ψ is a homomorphism of *-algebras. It is enough to show Ψ preserves multiplication and involution. For $s,t,u,v,z \in G$, $f_s \in C_s$, $g_u \in C_u$, and $x \in Q$,

$$\begin{aligned}
&\big(\Psi(f_s,t)\Psi(g_u,v)\big)(x,z)\\
&\quad= \sum_{r(y,w)=r(x,z)} \Psi(f_s,t)(y,w)\Psi(g_u,v)\big((y,w)^{-1}(x,z)\big)\\
&\quad= \sum_{\substack{r(y)=r(x)\\ c(y)w=c(x)z}} f_s(y)\Psi(g_u,v)\big((y^{-1},c(y)w)(x,z)\big) && \text{if } t=w \text{ (and 0 if not)}\\
&\quad= \sum_{r(y)=r(x)} f_s(y)\Psi(g_u,v)(y^{-1}x,z) && \text{if } t=c(y^{-1}x)z \text{ (and 0 if not)}\\
&\quad= \sum_{r(y)=r(x)} f_s(y)g_u(y^{-1}x) && \text{if } t=uz \text{ and } v=z \text{ (and 0 if not)}\\
&\quad= (f_sg_u)(x) && \text{if } t=uv \text{ and } v=z \text{ (and 0 if not)}\\
&\quad= \Psi(f_sg_u,v)(x,z) && \text{if } t=uv \text{ (and 0 if not)}\\
&\quad= \Psi\big((f_s,t)(g_u,v)\big)(x,z),
\end{aligned}$$

and for $s,t,u \in G$, $f_s \in C_s$, and $x \in Q$,

$$\begin{aligned}
\Psi(f_s,t)^*(x,u) &= \overline{\Psi(f_s,t)\big((x,u)^{-1}\big)}\\
&= \overline{\Psi(f_s,t)(x^{-1},c(x)u)}\\
&= \overline{f_s(x^{-1})} && \text{if } t=c(x)u \text{ (and 0 if not)}\\
&= f_s^*(x) && \text{if } st=u \text{ (and 0 if not)}\\
&= \Psi(f_s^*,st)(x,u)\\
&= \Psi\big((f_s,t^*\big)(x,u).
\end{aligned}$$

It follows that Ψ extends to an isomorphism of $C^*(Q) \times_\delta G$ onto $C^*(Q \times_c G)$, since these are enveloping C^*-algebras.

A straightforward calculation shows that Ψ intertwines the actions $\widehat{\delta}$ and β. $\square$

Remark 4.4. For Q amenable, the isomorphism of Theorem 4.3 can be deduced from [11, Theorem 3.2], although Masuda does everything spatially, with reduced coactions, reduced groupoid C^*-algebras, and crossed products represented on Hilbert space. To see this, note that the amenability of the skew product $Q \times_c G$ follows from that of Q by [18,

Proposition II.3.8], and that $C^*(Q) \times_\delta G$ is isomorphic to the spatial crossed product by the reduction of δ according to results in [14] and [17].

Corollary 4.5. *With the same hypotheses as Theorem* 4.3*,*

$$C^*(Q \times_c G) \times_{\beta,r} G \cong C^*(Q) \otimes \mathcal{K}(\ell^2(G)).$$

Proof. This follows immediately from Theorem 4.3 and Katayama's duality theorem [7, Theorem 8]. (See also the parenthetical remark in the proof of Corollary 2.5.) □

5. Skew-product groupoids: the full crossed product

In this section we prove a version of Corollary 4.5 for full crossed products, from which we can recover Proposition 3.7 of [8]. For this, we shall want to relate semidirect-product groupoids to crossed products. In general, if a discrete group G acts on a topological groupoid R, the *semidirect-product groupoid* $R \rtimes G$ is the product space $R \times G$ with the structure

$$(x,s)(y,t) = (x(s \cdot y), st) \quad \text{and} \quad (x,s)^{-1} = (s^{-1} \cdot x^{-1}, s^{-1})$$

whenever this makes sense. (This is readily seen to coincide with Renault's version in [18, Definition I.1.7].) If R is r-discrete and Hausdorff then so is $R \rtimes G$.

The following result is presumably folklore, but it never hurts to record groupoid facts.

Proposition 5.1. *Let G be a discrete group acting on an r-discrete Hausdorff groupoid R, and let β denote the associated action on $C^*(R)$. Then*

$$C^*(R \rtimes G) \cong C^*(R) \times_\beta G.$$

Proof. For $f \in C_c(R \rtimes G)$ define $\Phi(f) \in C_c(G, C_c(R)) \subseteq C_c(G, C^*(R))$ by

$$\Phi(f)(s)(x) = f(x,s) \qquad \text{for } s \in G, x \in R.$$

Then Φ is a *-homomorphism, since for $f, g \in C_c(R \rtimes G)$ we have

$$\begin{aligned}
\big(\Phi(f)\Phi(g)\big)(s)(x) &= \sum_t \big(\Phi(f)(t)\beta_t(\Phi(g)(t^{-1}s))\big)(x) \\
&= \sum_t \sum_{r(y)=r(x)} \Phi(f)(t)(y)\beta_t(\Phi(g)(t^{-1}s))(y^{-1}x) \\
&= \sum_{r(y)=r(x)} \sum_t f(y,t)g(t^{-1} \cdot (y^{-1}x), t^{-1}s) \\
&= \sum_{r(y,t)=r(x,s)} f(y,t)g((y,t)^{-1}(x,s)) \\
&= (fg)(x,s) = \Phi(fg)(s)(x)
\end{aligned}$$

and

$$\begin{aligned}
\Phi(f)^*(s)(x) &= \beta_s(\Phi(f)(s^{-1})^*)(x) = \Phi(f)(s^{-1})^*(s^{-1} \cdot x) = \overline{\Phi(f)(s^{-1})(s^{-1} \cdot x^{-1})} \\
&= \overline{f(s^{-1} \cdot x^{-1}, s^{-1})} = \overline{f((x,s)^{-1})} = f^*(x,s) = \Phi(f^*)(s)(x).
\end{aligned}$$

Since $C^*(R \rtimes G)$ is the enveloping C^*-algebra of $C_c(R \rtimes G)$, Φ extends uniquely to a homomorphism Φ of $C^*(R \rtimes G)$ into $C^*(R) \times_\beta G$.

To show Φ is an isomorphism, it suffices to find an inverse for the map $\Phi\colon C_c(R \rtimes G) \to C_c(G, C_c(R))$, since $C^*(R) \times_\beta G$ is the enveloping C^*-algebra of the *-algebra $C_c(G, C_c(R))$ (see, for example, [4, Lemma 3.3]). Given $f \in C_c(G, C_c(R))$ define $\Psi(f) \in C_c(R \rtimes G)$ by

$$\Psi(f)(x, s) = f(s)(x).$$

Since the support of $\Phi(f)$ in $R \times G$ is just the finite union of compact sets $\{s\} \times \operatorname{supp} f(s)$ as s runs through $\operatorname{supp} f$, $\Psi(f)$ has compact support. Moreover, it is obvious that Ψ is the required inverse for Φ at the level of C_c-functions. □

To show that the isomorphism Φ of Proposition 5.1 is suitably compatible with regular representations, we use two lemmas. For the first, consider an action β of a discrete group G on a C^*-algebra A. For any invariant closed ideal I of A, let $q\colon A \to A/I$ be the quotient map, and let $\tilde{\beta}$ be the associated action of G on A/I. Let $\operatorname{Ind} q\colon A \times_\beta G \to A/I \times_{\tilde{\beta},r} G$ be the unique homomorphism such that

$$\operatorname{Ind} q(f) = q \circ f \qquad \text{for } f \in C_c(G, A).$$

Then standard techniques from [19, Théorème 4.12] yield the following:

Lemma 5.2. *With the above assumptions and notation, there is a unique conditional expectation $P_{A\times G}$ of $A \times_\beta G$ onto A such that $P_{A\times G}(f) = f(e)$ for $f \in C_c(G, A)$. The composition $q \circ P_{A\times G}$ is a conditional expectation of $A \times_\beta G$ onto A/I such that for $b \in A \times_\beta G$,*

$$\operatorname{Ind} q(b) = 0 \quad \textit{if and only if} \quad q \circ P_{A\times G}(b^*b) = 0.$$

Now let G act on an r-discrete Hausdorff groupoid R, and let β denote the action of G on $C^*(R)$ such that

$$\beta_s(f)(x) = f(s^{-1} \cdot x) \qquad \text{for } f \in C_c(R),\ s \in G,\ x \in R.$$

Also let $\lambda_R\colon C^*(R) \to C^*_r(R)$ be the regular representation, viewed as a quotient map, and let P_R be the conditional expectation of $C^*(R)$ onto $C_0(R^0)$ such that

$$P_R(f) = f|_{R^0} \qquad \text{for } f \in C_c(R).$$

Then it follows from [18, Proposition II.4.8] that for $b \in C^*(R)$, $\lambda_R(b) = 0$ if and only if $P_R(b^*b) = 0$.

Lemma 5.3. *With the above assumptions and notation, the kernel of the regular representation λ_R is a β-invariant ideal of $C^*(R)$.*

Proof. It suffices to show that for $b \in C^*(R)$ and $s \in G$, $P_R(b) = 0$ if and only if $P_R \circ \beta_s(b) = 0$. Let $f \in C_c(R)$. Then

$$\|P_R \circ \beta_s(f)\| = \sup_{u\in R^0} |\beta_s(f)(u)| = \sup_{u\in R^0} \left|f(s^{-1} \cdot u)\right| = \sup_{u\in R^0} |f(u)| = \|P_R(f)\|.$$

Hence $\|P_R \circ \beta_s(b)\| = \|P_R(b)\|$ for all $b \in C^*(R)$, which proves the lemma. □

Note that Lemma 5.3 ensures that the map $\operatorname{Ind} \lambda_R$ is well-defined.

Proposition 5.4. *Let G be a discrete group acting on an r-discrete Hausdorff groupoid R, let β denote the associated action on $C^*(R)$, and let Φ be the isomorphism of Proposition* 5.1*. Then there is an isomorphism Φ_r such that the following diagram commutes:*

$$\begin{array}{ccc} C^*(R \rtimes G) & \xrightarrow{\Phi} & C^*(R) \times_\beta G \\ \Big\downarrow{\scriptstyle \lambda_{R\rtimes G}} & & \Big\downarrow{\scriptstyle \operatorname{Ind}\lambda_R} \\ C^*_r(R \rtimes G) & \xrightarrow[\Phi_r]{} & C^*_r(R) \times_{\beta,r} G. \end{array}$$

Proof. We need only show that $\ker\big(\operatorname{Ind}\lambda_R \circ \Phi\big) = \ker \lambda_{R\times G}$. Take a positive element b of $C^*(R \rtimes G)$. By Lemma 5.2, $\operatorname{Ind}\lambda_R \circ \Phi(b) = 0$ if and only if $\lambda_R \circ P_{C^*(R)\times G} \circ \Phi(b) = 0$ (because $\Phi(b)$ is positive in $C^*(R) \times_\beta G$), so that $\operatorname{Ind}\lambda_R \circ \Phi(b) = 0$ if and only if $P_R \circ P_{C^*(R)\times G} \circ \Phi(b) = 0$ (because $P_{C^*(R)\times G} \circ \Phi(b)$ is positive in $C^*(R)$). On the other hand, $\lambda_{R\rtimes G}(b) = 0$ if and only if $P_{R\rtimes G}(b) = 0$. Thus, it suffices to show that for all $b \in C^*(R \rtimes G)$,

$$\|P_{R\rtimes G}(b)\| = \big\|P_R \circ P_{C^*(R)\times G} \circ \Phi(b)\big\|,$$

and for this it suffices to take $b \in C_c(R \rtimes G)$:

$$\begin{aligned}\big\|P_R \circ P_{C^*(R)\times G} \circ \Phi(b)\big\| &= \big\|P_{C^*(R)\times G} \circ \Phi(b)|_{R^0}\big\| = \|\Phi(b)(e)|_{R^0}\| = \sup_{u\in R^0} |\Phi(b)(e)(u)| \\ &= \sup_{u\in R^0} |b(u,e)| = \big\|b|_{(R^0\times\{e\})}\big\| = \big\|b|_{(R\rtimes G)^0}\big\| = \|P_{R\rtimes G}(b)\|.\end{aligned}$$

This completes the proof. □

We write $Q \times_c G \rtimes G$ for the semidirect product of G acting on $Q \times_c G$, and we write the elements as triples.

Proposition 5.5. *Let c be a continuous homomorphism of an r-discrete Hausdorff groupoid Q into a discrete group G, and let G act on the skew product $Q \times_c G$ as in Equation* (4.1)*. Then the semidirect-product groupoid $Q \times_c G \rtimes G$ is equivalent to Q.*

Proof. We will show that the space $Q \times_c G$ implements a groupoid equivalence (in the sense of [12, Definition 2.1]) between $Q \times_c G \rtimes G$ (acting on the left) and Q (acting on the right). For the right action we need a continuous open surjection σ from $Q \times_c G$ onto the unit space of Q. For $(x, s) \in Q \times_c G$ define $\sigma(x, s) = s(x)$. Then σ is a continuous and open surjection onto Q^0. Now put

$$(Q \times_c G) * Q = \{((x, s), y) \in (Q \times_c G) \times Q \mid \sigma(x, s) = r(y)\},$$

and define a map $((x, s), y) \mapsto (x, s)y$ from $(Q \times_c G) * Q$ to $Q \times_c G$ by

$$(x, s)y = (xy, c(y)^{-1}s).$$

The continuity and algebraic properties of this map are easily checked, so we have a right action of Q on the space $Q \times_c G$. For the left action we need a continuous and open surjection ρ from $Q \times_c G$ onto the unit space of $Q \times_c G \rtimes G$. Note that this unit space is $Q^0 \times G \times \{e\}$, and the range and source maps in $Q \times_c G \rtimes G$ are given by

$$r(x, s, t) = (r(x), c(x)s, e) \quad \text{and} \quad s(x, s, t) = (s(x), s, e).$$

For $(x,s) \in Q \times_c G$ define $\rho(x,s) = (r(x), c(x)s, e)$. Then ρ is a continuous surjection onto $Q^0 \times G \times \{e\}$, and ρ is open since r is and G is discrete. Now put

$$\begin{aligned}(Q \times_c G \rtimes G) &* (Q \times_c G) \\ &= \{((x,s,t),(y,r)) \in (Q \times_c G \rtimes G) \times (Q \times_c G) \mid s(x,s,t) = \rho(y,r)\},\end{aligned}$$

and define a map $((x,s,t),(y,r)) \mapsto (x,s,t)(y,r)$ from $(Q \times_c G \rtimes G) * (Q \times_c G)$ by

$$(x,s,t)(y,r) = (xy, rt^{-1}).$$

The continuity and algebraic properties of this map are also easily checked, so we have a left action of $Q \times_c G \rtimes G$ on the space $Q \times_c G$.

Next we must show that both actions are free and proper, and that they commute. If $(x,s,t)(y,r) = (y,r)$, then $xy = y$ and $rt^{-1} = r$, so x is a unit and $t = e$, hence (x,s,t) is a unit; thus the left action is free. For properness of the left action, it is enough to show that if L is compact in Q and F is finite in G, then there is some compact set in $(Q \times_c G \rtimes G) * (Q \times_c G)$ containing all pairs $((x,s,t),(y,r))$ for which

$$((x,s,t)(y,r),(y,r)) \in (L \times F) \times (L \times F).$$

But the above condition forces $x \in LL^{-1}$, $s \in c(L)FF^{-1}F$, $t \in F^{-1}F$, $y \in L$, and $r \in F$, so the left action is proper. Freeness and properness of the right action is checked similarly (but more easily), and it is straightforward to verify that the actions commute.

To show $Q \times_c G$ is a $(Q \times_c G \rtimes G)$-Q equivalence, it remains to verify that the map ρ factors through a bijection of $(Q \times_c G)/Q$ onto $(Q \times_c G \rtimes G)^0$, and similarly that the map σ factors through a bijection of $(Q \times_c G \rtimes G) \setminus (Q \times_c G)$ onto Q^0. Since ρ and σ are surjective and the actions commute, it suffices to show that $\rho(x,s) = \rho(y,t)$ implies $(x,s) \in (y,t)Q$, and $\sigma(x,s) = \sigma(y,t)$ implies $(x,s) \in (Q \times_c G \rtimes G)(y,t)$. For the first, if $\rho(x,s) = \rho(y,t)$ then $r(x) = r(y)$ and $c(x)s = c(y)t$. Put $z = y^{-1}x$; then $x = yz$ and $c(z)^{-1}t = c(x)^{-1}c(y)t = s$, so $(x,s) = (y,t)z$. For the second, if $\sigma(x,s) = \sigma(y,t)$ then $s(x) = s(y)$. Put $z = xy^{-1}, r = c(y)s$, and $q = s^{-1}t$; then $x = zy$, $s = tq^{-1}$, and $c(y)tq^{-1} = c(y)s = r$, so $(x,s) = (z,r,q)(y,t)$. □

Proposition 5.6. *Let c be a continuous homomorphism of an r-discrete Hausdorff groupoid Q into a discrete group G, and suppose Q is amenable. Then the action β of G on $C^*(Q \times_c G)$ defined by Equation* (4.2) *is amenable in the sense that the regular representation of $C^*(Q \times_c G) \times_\beta G$ is faithful.*

Proof. First note that [2, Proposition 6.1.7], for example, implies that the full and reduced C^*-algebras of an amenable groupoid coincide. Since Q is amenable so is the skew product $Q \times_c G$, by [18, Proposition II.3.8]; hence $C^*(Q \times_c G) = C^*_r(Q \times_c G)$ and $\operatorname{Ind} \lambda_{Q \times_c G}$ is just the regular representation $\lambda_{C^*(Q \times_c G) \times G}$. The semidirect-product groupoid $Q \times_c G \rtimes G$ is also amenable, by Proposition 5.5, since groupoid equivalence preserves amenability ([2,

Theorem 2.2.13]). Thus, Proposition 5.4 gives a commutative diagram

$$\begin{tikzcd} C^*(Q \times_c G \rtimes G) \arrow[r, "\Phi"] \arrow[dr, "\Phi_r"'] & C^*(Q \times_c G) \times_\beta G \arrow[d, "\lambda_{C^*(Q\times_c G)\times G}"] \\ & C^*(Q \times_c G) \times_{\beta,r} G \end{tikzcd}$$

in which Φ and Φ_r are isomorphisms. This proves the proposition. □

Remark 5.7. The above result could also be proved using [1, Théorème 4.5 and Proposition 4.8], since both $C^*(Q \times_c G)$ and $C^*(Q \times_c G) \times_{\beta,r} G$ are nuclear (by [2, Proposition 3.3.5 and Corollary 6.2.14] and [18, Proposition II.3.8]).

Lemma 5.8. *Let c be a continuous homomorphism of an r-discrete Hausdorff groupoid Q into a discrete group G, and put $N = c^{-1}(e)$. Assume that Q is amenable. Then the canonical map $i\colon C^*(N) \to C^*(Q)$ is faithful.*

Proof. Since Q is amenable, so is N [2, Proposition 5.1.1]. Let $P_Q\colon C^*(Q) \to C_0(Q^0)$ denote the unique conditional expectation extending the map $f \mapsto f|_{Q^0}$ at the level of C_c-functions. Since Q is amenable, the regular representation of $C^*(Q)$ onto $C^*_r(Q)$ is faithful [2, Proposition 6.1.7]. By [18, Proposition II.4.8], this implies P_Q is faithful in the sense that $a \in C^*(Q)$ and $P_Q(a^*a) = 0$ imply $a = 0$, and similarly for P_N (Renault assumes Q is principal, but this is not used in showing his conditional expectation is faithful on the reduced C^*-algebra $C^*_r(Q)$). It is easy to see by checking elements of $C_c(N)$ that $P_N = P_Q \circ i$. If $a \in \ker i$ then so is a^*a, thus $P_N(a^*a) = 0$, so $a^*a = 0$ since N is amenable, hence $a = 0$. □

It is easy to check that roughly the same argument as above would work if we only assume N itself is amenable.

Theorem 5.9. *Let c be a continuous homomorphism of an amenable r-discrete Hausdorff groupoid Q into a discrete group G, and let β be the action of Equation* (4.2)*. Then*

$$C^*(Q \times_c G) \times_\beta G \cong C^*(Q) \otimes \mathcal{K}(\ell^2(G)).$$

Proof. Lemma 5.8 ensures that the hypotheses of Corollary 4.5 are satisfied, which gives

$$C^*(Q \times_c G) \times_{\beta,r} G \cong C^*(Q) \otimes \mathcal{K}(\ell^2(G)).$$

The theorem now follows from Proposition 5.6. □

6. Embedding $C^*(N)$ in $C^*(Q)$

In this section we fulfill the promise made just before Theorem 4.3 by showing the map $i\colon C^*(N) \to C^*(Q)$ is faithful when Q is second countable. But first we need the following elementary lemma, which we could not find in the literature.

Lemma 6.1. *Let Q be an r-discrete Hausdorff groupoid, and let π be a *-homomorphism from $C_c(Q)$ to the *-algebra of adjointable linear operators on an inner product space $\mathcal{H}$. Then for all $a \in C_c(Q)$, the operator $\pi(a)$ is bounded and $\|\pi(a)\| \leq \|a\|$, where $C_c(Q)$ is given the largest C^*-norm.*

Proof. Let $a \in C_c(Q)$. Since $C_c(Q)$ has the largest C^*-norm, it suffices to show $\pi(a)$ is bounded. Choose open bisections ("Q-sets", in Renault's terminology) $\{U_i\}_1^n$ of Q such that $\operatorname{supp} a \subseteq \bigcup_1^n U_i$, and a partition of unity $\{\phi_i\}_1^n$ subordinate to the open cover $\{U_i\}_1^n$ of $\operatorname{supp} a$. Then $a = \sum_1^n a\phi_i$, and $\operatorname{supp} a\phi_i \subseteq U_i$. Conclusion: without loss of generality there exists an open bisection U of Q such that $\operatorname{supp} a \subseteq U$. Then $\operatorname{supp} a^*a \subseteq U^{-1}U$, a relatively compact subset of the unit space Q^0. Choose an open set $V \subseteq Q^0$ such that $\overline{U^{-1}U} \subseteq V$ and $\overline{V}$ is compact. Then $a^*a \in C_0(V)$, which is a C^*-subalgebra of the commutative *-subalgebra $C_c(Q^0)$ of $C_c(Q)$. Since π restricts to a *-homomorphism from $C_0(V)$ to the adjointable linear operators on $\mathcal{H}$, $\pi(a^*a)$ is bounded. Since $\pi(a)^*\pi(a) = \pi(a^*a)$, $\pi(a)$ is bounded as well. □

Theorem 6.2. *Let c be a continuous homomorphism of an r-discrete Hausdorff groupoid Q into a discrete group G, and put $N = c^{-1}(e)$. Assume that Q is second countable. Then the canonical map $i\colon C^*(N) \to C^*(Q)$ is faithful.*

Proof. For notational convenience, throughout this proof we suppress the map i, and identify $C_c(N)$ and $C^*(N)$ with their images in $C^*(Q)$. Our strategy is to find a C^*-seminorm on $C_c(Q)$ which restricts to the greatest C^*-norm on $C_c(N)$. This suffices, for then *a fortiori* the greatest C^*-norm on $C_c(Q)$ restricts to the greatest C^*-norm on $C_c(N)$, which is what we need to prove.

To get this C^*-seminorm on $C_c(Q)$, we make $C_c(Q)$ into a pre-Hilbert $C_c(N)$-module, and show that by left multiplication $C_c(Q)$ acts by bounded adjointable operators. We do this by showing that the space Q implements a groupoid equivalence in the sense of [12, Definition 2.1] between N (acting on the right) and a suitable groupoid H (acting on the left); then the construction of [12] shows that $C_c(Q)$ is a pre-imprimitivity bimodule, and in particular a right pre-Hilbert $C_c(N)$-module.

We define

$$H = \{(x, c(y)) \mid x, y \in Q, s(x) = r(y)\},$$

which is a subgroupoid of the skew product $Q \times_c G$. We claim that H is open in $Q \times_c G$. Let $(x, t) \in H$. There exists $y \in Q^{s(x)}$ such that $c(y) = t$, and then there exists a neighborhood V of y such that $c(V) \subseteq \{s\}$. Then $r(V)$ is a neighborhood of $r(y) = s(x)$, so there exists a neighborhood U of x such that $s(U) \subseteq r(V)$. By construction, for all $z \in U$ there exists $w \in V$ such that $r(w) = s(z)$, and then $c(w) = t$. Therefore, the open subset $U \times \{t\}$ of $Q \times G$ is contained in H, so (x, t) is an interior point of H. This proves the claim. Since the restriction of a Haar system to an open subgroupoid gives a Haar system, counting measures give a Haar system on H. Since Q is second countable, the image of the homomorphism c in G is countable, hence the groupoid H is second countable. Since the skew-product groupoid $Q \times_c G$ is r-discrete, so is the open subgroupoid H.

The subgroupoid N acts on the right of Q by multiplication. We want to define a left action of the groupoid H on the space Q. For this we need a continuous and open

surjection ρ from Q onto the unit space of H. We have $H^0 = \{(u,t) \in Q^0 \times G \mid t \in c(Q^u)\}$, and the range and source maps in H are given by

$$r(x,t) = (r(x), c(x)t) \quad \text{and} \quad s(x,t) = (s(x), t).$$

For $y \in Q$ define

$$\rho(y) = (r(y), c(y)).$$

Then ρ is a continuous surjection onto H^0, and is open since r is and G is discrete. Now put

$$H * Q = \{((x,t), y) \in H \times Q \mid s(x,t) = \rho(y)\},$$

and define a map $((x,t),y) \mapsto (x,t)y$ from $H * Q$ to Q by

$$(x,t)y = xy.$$

The continuity and algebraic properties of this map are easily checked, so we have an action of H on Q.

Next we must show that both actions are free and proper, and that the actions commute. Since $(x,t)y = y$ implies that x, hence (x,t), is a unit, the left action is free. For properness of the left action, let K be a compact subset of $Q \times Q$. We must show that the inverse image of K under the map $((x,t),y) \mapsto ((x,t)y, y)$ from $H * Q$ to $Q \times Q$ is compact. Without loss of generality suppose $K = L \times L$ for some compact subset L of Q. For all $((x,t),y) \in H * Q$, if $((x,t)y, y) \in L \times L$ then $x \in LL^{-1}$, $t \in c(L)$, and $y \in L$, so the inverse image of K is contained in

$$(LL^{-1} \times c(L)) * L,$$

which is compact in $(Q \times_c G) * Q$. It is easier to see that the right N-action is free and proper, and straightforward to check that the actions commute.

To show Q is an H–N equivalence, it remains to verify that the map ρ factors through a bijection of Q/N onto H^0, and similarly that the map $s\colon Q \to H^0$ factors through a bijection of $H\backslash Q$ onto N^0. Since ρ and s are surjective and the actions commute, it suffices to show that $\rho(y) = \rho(z)$ implies $z \in yN$ and $s(y) = s(z)$ implies $z \in Hy$. For the first, if $\rho(y) = \rho(z)$ then $r(y) = y(z)$ and $c(y) = c(z)$. Put $n = y^{-1}z$. Then $c(n) = c(y)^{-1}c(z) = e$, so $n \in N$, and $z = yn$. For the second, if $s(y) = s(z)$, put $x = zy^{-1}$. Then $(x, c(z)) \in H$ and $z = xy = (x, c(z))y$.

Now the theory of [12] tells us $C_c(Q)$ becomes a pre-Hilbert $C_c(N)$-module, where $C_c(N)$ is given the C^*-norm from $C^*(N)$. From the formulas in [12] the right module multiplication is given by

$$ac(x) = \sum_{r(n)=s(x)} a(xn)c(n^{-1}),$$

where $a \in C_c(Q)$ and $c \in C_c(N)$, and the inner product is

$$\langle a, b\rangle_{C_c(N)}(n) = \sum_{r(x,s)=\rho(y)} \overline{a((x,s)^{-1}y)}\, b((x,s)^{-1}yn), \tag{6.1}$$

where $a, b \in C_c(Q)$ and y is any element of Q with $s(y) = r(n)$. The right module action is just right multiplication by the subalgebra $C_c(N)$ inside the algebra $C_c(Q)$. The inner

product also simplifies in our situation: let $a, b \in C_c(Q)$, and write $a = \sum_{t\in G} a_t$ and $b = \sum_{t\in G} b_t$ with $a_t, b_t \in C_t = \{f \in C_c(Q) \mid \operatorname{supp} f \subseteq c^{-1}(t)\}$. We claim that

$$\langle a, b\rangle_{C_c(N)} = \sum_{t\in G} a_t^* b_t.$$

Of course, we are identifying $a_t^* b_t$ with $a_t^* b_t|_N$, but this causes no harm since $a_t^* b_t$ is supported in N. In Equation (6.1) we can take $y = r(n)$, so that $\rho(y) = (r(n), e)$. Then the condition $r(x, s) = \rho(y)$ becomes $r(x) = r(n)$ and $c(x)s = e$, so that

$$\begin{aligned}
\langle a, b\rangle_{C_c(N)}(n) &= \sum_{\substack{r(x)=r(n)\\ c(x)s=e}} \overline{a((x^{-1}, c(x)s)r(n))}\, b((x^{-1}, c(x)s)n) \\
&= \sum_{\substack{r(x)=r(n)\\ c(x)s=e}} \overline{a(x^{-1})}\, b(x^{-1}n) \\
&= \sum_{\substack{r(x)=r(n)\\ c(x)s=e}} a^*(x) b(x^{-1}n) \\
&= \sum_{t,r\in G} \sum_{\substack{r(x)=r(n)\\ c(x)s=e}} a_t^*(x) b_r(x^{-1}n) \\
&= \sum_{t\in G} \sum_{r(x)=r(n)} a_t^*(x) b_t(x^{-1}n).
\end{aligned}$$

Since in this last expression we need only consider terms with $c(x) = t^{-1}$ and $c(x^{-1}n) = r$, which forces $t = r$, and then $s = t$ in the inner sum, this gives

$$\langle a, b\rangle_{C_c(N)}(n) = \sum_{t\in G} a_t^* b_t(n).$$

This proves the claim.

Now we show that for fixed $a \in C_c(Q)$, the map $b \mapsto ab$ is a bounded adjointable operator on the pre-Hilbert $C_c(N)$-module $C_c(Q)$, with adjoint $b \mapsto a^*b$. This will give a representation of $C_c(Q)$ in $\mathcal{L}_{C_c(N)}(C_c(Q))$, hence a C^*-seminorm on $C_c(Q)$.

We first handle the adjointability. Without loss of generality let $a \in C_s$ and take $b = \sum_t b_t, c = \sum_t c_t \in C_c(Q)$ with $b_t, c_t \in C_t$. Then

$$\begin{aligned}
\langle ab, c\rangle_{C_c(N)} &= \langle \textstyle\sum_t ab_t, \sum_t c_t\rangle_{C_c(N)} = \sum_t (ab_t)^* c_{st} && \text{(since } ab_t \in C_{st}\text{)} \\
&= \textstyle\sum_t b_t^* a^* c_{st} = \langle \sum_t b_t, \sum_t a^* c_{st}\rangle_{C_c(N)} && \text{(since } a^*c_{st} \in C_t\text{)} \\
&= \langle b, a^* \textstyle\sum_t c_{st}\rangle_{C_c(N)} = \langle b, a^* c\rangle_{C_c(N)}.
\end{aligned}$$

For the boundedness, let ω be a state on $C^*(N)$, and let $\langle \cdot, \cdot\rangle_\omega = \omega(\langle \cdot, \cdot\rangle_{C_c(N)})$ be the associated semi-inner product on $C_c(Q)$. Let $\mathcal{H}$ be the corresponding inner product space, and let $\Theta\colon C_c(Q) \to \mathcal{H}$ be the quotient map. Then left multiplication defines a *-homomorphism π from $C_c(Q)$ to the *-algebra of adjointable linear operators on $\mathcal{H}$ via

$\pi(a)\Theta(b) = \Theta(ab)$. As we show in the general lemma below, for all $a \in C_c(Q)$, the operator $\pi(a)$ is bounded and $\|\pi(a)\| \leq \|a\|$. Hence, for all $a \in C_c(Q)$ and $b \in C_c(Q)$,

$$\omega\big(\langle ab, ab\rangle_{C_c(N)}\big) = \langle \pi(a)\Theta(b), \pi(a)\Theta(b)\rangle_\omega$$
$$\leq \|\pi(a)\|^2 \langle \Theta(b), \Theta(b)\rangle_\omega \leq \|a\|^2 \omega\big(\langle b, b\rangle_{C_c(N)}\big).$$

Since the state ω was arbitrary,

$$\langle ab, ab\rangle_{C_c(N)} \leq \|a\|^2 \langle b, b\rangle_{C_c(N)},$$

as required.

We can now define a C^*-seminorm $\|\cdot\|_*$ on $C_c(Q)$ by letting $\|a\|_*$ be the norm of the operator $b \mapsto ab$ in $\mathcal{L}_{C_c(N)}(C_c(Q))$. To finish, we need to know that for $a \in C_c(N)$ the norm $\|a\|_*$ agrees with the greatest C^*-norm $\|a\|$, and it suffices to show $\|a\| \leq \|a\|_*$:

$$\|a\|^2 = \|a^*a\| \leq \|a^*\|_* \|a\| = \|a\|_* \|a\|,$$

since a^*a is a value of the operator $c \mapsto a^*c$, and then canceling $\|a\|$ gives the desired inequality. This completes the proof. □

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